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EXPANDING GRAPHS CONTAIN ALL SMALL TREES

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The assertion of the title is formulated and proved. The result is then used to construct graphs with a linear number of edges that, even after the deletion of almost all of their edges or almost all of their vertices, continue to contain all small trees.

1. Introduction

If H is an undirected graph, $V(H)$ will denote its set of vertices and $E(H)$ will denote its set of edges. If $X \subseteq V(H)$, $\Gamma_H(X)$ will denote the set of neighbors in H of vertices in X. If X is a set, $|X|$ will denote its cardinality.

The following theorem, which is implicit in a result of Pósa [5], has been given an elegant proof by Lovász ([3], Ch. 10. Problem 20).

Theorem 0. *If H is a non-empty graph such that, for each $X \subseteq V(H)$ with $|X| \leqq n$,*

$$|\Gamma_H(X) \setminus X| \geqq 2|X| - 1,$$

then H contains a path with $3n-2$ vertices.

Using Theorem 0, Beck [2] proved an upper bound of the form $O(n)$ for the minimum possible number of edges in graphs that, even after the deletion of half their edges, continue to contain a path with n vertices; Alon and Chung [1] have given an explicit construction for such graphs.

The main result of this paper is the following theorem.

Theorem 1. *If H is a non-empty graph such that, for every $X \subseteq V(H)$ with $|X| \leqq 2n-2$,*

$$|\Gamma_H(X)| \geqq (d+1)|X|,$$

then H contains every tree with n vertices and maximum degree at most d.

Since a path with n vertices is the unique tree with n vertices and maximum degree 2, Theorem 1 generalizes the essence of Theorem 0 from paths to trees. Using Theorem 1, the arguments of Beck [2] and Alon and Chung [1], and a recent result of Lubotzky, Phillips and Sarnak [4], we prove the following theorem.

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Theorem 2. *Let $\delta>0$ and d be fixed. For every n there is a graph F with $O(n)$ edges that, even after deletion of all but $\delta|E(F)|$ edges, continues to contain every tree with n vertices and maximum degree at most d.*

For $\delta=1/2$, Beck [2] proved an upper bound, without an explicit construction, of the form $O\big(n(\log n)^{12}\big)$. We shall also prove the following theorem.

Theorem 3. *Let $\varepsilon>0$ and d be fixed. For every n there is a graph F with $O(n)$ edges that, even after deletion of all but $\varepsilon|V(F)|$ vertices, continues to contain every tree with n vertices and maximum degree at most d.*

2. Proof of Theorem 1

If T is a tree and H is a graph, a map $f\colon V(T)\to V(H)$ will be called an *embedding* of T in H if it is injective and $f(v)$ and $f(w)$ are adjacent in H whenever v and w are adjacent in T. A tree T will be called *small* if it has at most n vertices and maximum degree at most d. (The parameters n and d will remain fixed throughout this proof.) A graph H will be called *expanding* if, for every $X\subseteq V(H)$ with $|X|\leqq 2n-2$,

$$|\Gamma_H(X)| \geqq (d+1)|X|.$$

Our goal is to show that if T is small and H is non-empty and expanding, then there is an embedding of T in H. To achieve this we shall define a class of "good" embeddings. We shall then show that this class has the following two properties.

Property 1. If T consists of a single vertex and H is a non-empty expanding graph, then there is a good embedding of T in H.

Property 2. If T is a small tree and S is a subtree of T obtained by deleting a leaf and the edge incident with it, then any good embedding of S in an expanding graph H can be extended to a good embedding of T in H.

When this has been done, it will follow by induction on $|T|$ that, if T is a small tree and H is a non-empty expanding graph, then there is a good embedding of T in H. If $|V(T)|=1$, this follows from Property 1. If $|V(T)|\geqq 2$, let S be any tree obtained from T by deleting a leaf and the edge incident with it. By inductive hypothesis, there is a good embedding of S in H, and by Property 2, this can be extended to a good embedding of T in H. This completes the induction and the proof of Theorem 1.

To define good embeddings, we shall need some auxiliary definitions. Let f be an embedding of a tree T in a graph H. If $X\subseteq V(H)$, we shall define the *assets* $A_f(X)$ of X under f to be $\big|\Gamma_H(X)\setminus f\big(V(T)\big)\big|$. If $x\in V(H)$, we shall let $J_f(x)$ denote the degree of $f^{-1}(x)$ in T if $x\in f(T)$, and 0 otherwise. We shall let $B_f(x)$ denote $d-J_f(x)$. If $X\subseteq V(H)$, we shall define the *liabilities* $B_f(X)$ of X under f to be $\sum_{x\in X} B_f(x)$, and the *balance* $C_f(X)$ of X under f to be $A_f(X)-B_f(X)$. A set $X\subseteq V(H)$ will be called *solvent* under f if $C_f(X)\geqq 0$, *critical* under f if $C_f(X)=0$, and *bankrupt* under f if $C_f(X)<0$. Finally, we arrive at the key definition. An embedding f of a tree T in a graph H is *good* if every $X\subseteq V(H)$ with $|X|\leqq 2n-2$ is solvent. It remains to prove Properties 1 and 2.

To prove Property 1, suppose that T is a tree consisting of a single vertex and that H is a non-empty expanding graph. Since H is non-empty, there is an embedding f of T in H. We shall show that f is good. Suppose $X \subseteq V(H)$ and $|X| \leqq 2n-2$. Since H is expanding,

$$|\Gamma_H(X)| \geqq (d+1)|X|.$$

Since $f(V(T))$ consists of a single vertex,

$$A_f(X) = |\Gamma_H(X) \setminus f(V(T))| \geqq d|X|.$$

On the other hand, $J_f(x)=0$, so $B_f(x)=d$ and

$$B_f(X) = d|X|.$$

Thus X is solvent, which completes the proof of Property 1.

To prove Property 2, suppose that T is a small tree, that S is a subtree obtained from T by deleting a leaf v and the incident edge $\{v, w\}$, and that f is a good embedding of S in an expanding graph H. Let G denote the set of embeddings of T in H that are extensions of f. Let $Y=\Gamma_H(\{f(w)\}) \setminus f(V(S))$. The map $g \mapsto g(v)$ is clearly a bijection between G and Y. We must show that some $g \in G$ is good.

Suppose, to obtain a contradiction, that no $g \in G$ is good. Then for every $g \in G$, there is a set $X_g \subseteq V(H)$ with $|X_g| \leqq 2n-2$ and X_g bankrupt under g. On the other hand, since f is a good embedding, we have X_g solvent under f for all $g \in G$. Since $g(V(T))=f(V(S)) \cup \{g(v)\}$, we have

$$A_g(X) = A_f(X) - I(g(v), \Gamma_H(X)),$$

where $I(z, Z)$ is 1 if $z \in Z$ and 0 otherwise. Since $E(T)=E(S) \cup \{\{v, w\}\}$, we have

$$B_g(X) = B_f(X) - I(f(w), X) - I(g(v), X).$$

Thus,

$$C_g(X) = C_f(X) - I(g(v), \Gamma_H(X)) + I(f(w), X) + I(g(v), X).$$

For each $g \in G$ we have $C_g(X_g) < 0$ and $C_f(X_g) \geqq 0$, so we must have $C_f(X_g)=0$, $I(g(v), \Gamma_H(X))=1$, $I(f(w), X)=0$ and $I(g(v), X)=0$. Thus for every $g \in G$, we have X_g critical, $g(v) \in \Gamma_H(X_g)$, $f(w) \notin X_g$ and $g(v) \notin X_g$. To proceed further, we shall need some lemmas.

Lemma 1.1. *If $X \subseteq V(H)$ is critical under f and $|X| \leqq 2n-2$, then $|X| \leqq n-1$.*

Proof. Since H is an expanding graph and $|X| \leqq 2n-2$,

$$|\Gamma_H(X)| \geqq (d+1)|X|.$$

Since T is a small tree and S is obtained by deleting a leaf, $|V(S)| \leqq n-1$, and thus

$$A_f(X) = |\Gamma_H(X) \setminus f(V(S))| \geqq (d+1)|X| - (n-1).$$

On the other hand, $J_f(x) \geqq 0$, so $B_f(x) \leqq d$ and

$$B_f(X) \leqq d|X|.$$

Since X is critical under f, $A_f(X)=B_f(X)$, so $|X| \leqq n-1$. ∎

Lemma 1.2. *The balance $C_f(\cdot)$ is submodular; that is*

$$C_f(X\cup Y)+C_f(X\cap Y) \leqq C_f(X)+C_f(Y).$$

Proof. Since $B_f(X)$ is defined by a sum over vertices in X, $B_f(\cdot)$ is *modular*; that is

$$B_f(X\cup Y)+B_f(X\cap Y) = B_f(X)+B_f(Y).$$

Thus it suffices to show that $A_f(\cdot)$ is submodular; but this is an immediate consequence of the relations $\Gamma_H(X\cup Y)=\Gamma_H(X)\cup\Gamma_H(Y)$ and $\Gamma_H(X\cap Y)\subseteqq\Gamma_H(X)\cap \cap\Gamma_H(Y)$. ∎

Lemma 1.3. *If $X, Y\subseteqq V(H)$ are critical under f and $|X|, |Y|\leqq n-1$, then $X\cup Y$ is critical under f and $|X\cup Y|\leqq n-1$.*

Proof. Since f is a good embedding and $|X\cup Y|, |X\cap Y|\leqq 2n-2$, $C_f(X\cup Y)$, $C_f(X\cap Y)\geqq 0$. Since X and Y are critical under f, $C_f(X)$, $C_f(Y)=0$. Thus Lemma 1.2 implies $C_f(X\cup Y)\leqq 0$, whence $C_f(X\cup Y)=0$ and $X\cup Y$ is critical. Lemma 1.1 now implies $|X\cup Y|\leqq n-1$. ∎

We now resume the proof of Property 2. For every $g\in G$, X_g is critical under f and $|X_g|\leqq 2n-2$; thus by Lemma 1.1, $|X_g|\leqq n-1$. We now claim that the set

$$X^* = \bigcup_{g\in G} X_g$$

is critical under f and that $|X^*|\leqq n-1$. If $|G|=0$, this is trivial; if $|G|\geqq 1$, it follows by induction using Lemma 1.3. Now consider $X'=X^*\cup\{f(w)\}$. Since f is good and $|X'|\leqq n$, X' is solvent under f. Since $g(v)\in\Gamma_H(X_g)$ for every $g\in G$, $Y\subseteqq\Gamma_H(X^*)$, which implies

$$A_f(X') = A_f(X^*).$$

Since $f(w)\notin X_g$ for every $g\in G$, $f(w)\notin X^*$, which implies

$$B_f(X') = B_f(X^*)+B_f(f(w)).$$

Since T is small, w has degree at most d in T and thus degree at most $d-1$ in S, so $J_f(f(w))\leqq d-1$ and $B_f(f(w))\geqq 1$. Thus $B_f(X')>B_f(X^*)$, so $C_f(X')<C_f(X^*)$. Since X^* is critical under f, X' is bankrupt under f. This contradicts the fact that X' is solvent under f, completing the proof of Property 2 and of Theorem 1.

3. Proof of Theorem 2

Let us say that a graph F with m vertices is (k, l)-*spectral* if

(A) every vertex of F has degree k, so that the adjacency matrix of F (which, by abuse of notation, we shall also denote by F), has an eigenvalue $\lambda_0(F)=k$ corresponding to an eigenvector e that assigns the value 1 to every vertex of F; and

(B) each of the other eigenvalues $\lambda_j(F)$, $1\leqq j\leqq m-1$, of F satisfies $|\lambda_j(F)|\leqq l$.

Lubotzky, Phillips and Sarnak [4] have shown that if p and q are primes congruent to 1 modulo 4, with p a quadratic non-residue modulo q, then there is an explicitly construced graph F that

(1) has $m=q(q^2-1)/2$ vertices;
(2) is $(k, 2(k-1)^{1/2})$-spectral, where $k=p+1$; and
(3) has girth at least $(2/3)\log_{k-1} m$.

If F is a graph and $X \subseteq V(F)$, $\Theta_F(X)$ will denote the set of edges of F having both ends in X, $\Phi_F(X)$ will denote the set of edges of F having at least one end in X, and $\Psi_F(X)$ will denote the set of edges of F having exactly one end in X. We shall need the following lemma.

Lemma 2.1. *If a graph F has m vertices and is (k, l)-spectral, and if $X \subseteq V(F)$, then*

$$|\Theta_F(X)| \leqq k|X|^2/2m + l|X|/2.$$

Proof. Define the function g on $V(F)$ by $g(x)=1-|X|/m$ if $x \in X$ and $g(x)= -|X|/m$ otherwise. Then

$$|\Psi_F(X)| = \frac{1}{2} \sum_{\{x,y\} \in E(F)} (g(x)-g(y))^2.$$

Expanding the sum yields

$$\begin{aligned} |\Psi_F(X)| &= k \sum_{x \in V(F)} g(x)^2 - \sum_{\{x,y\} \in E(F)} g(x)g(y) \\ &= k\langle g, g\rangle - \langle g, Fg\rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes inner product. Since g is orthogonal to the eigenvector e, $\langle g, Fg\rangle \leqq l\langle g, g\rangle$. Thus $|\Psi_F(X)| \geqq (k-l)\langle g, g\rangle = (k-l)|X|(1-|X|/m)$. Since $|\Theta_F(X)| = (k|X|-|\Psi_F(X)|)/2$, $|\Theta_F(X)| \leqq k|X|^2/2m + l|X|(1-|X|/m)/2 \leqq k|X|^2/2m + l|X|/2$. ∎

Let $\delta > 0$, d and n be given. Assign p and q the smallest appropriate values for which the following conditions are satisfied.

Condition 1. $k \geqq 16(d+2)^2/\delta^2$.

Condition 2. $m \geqq 2(d+2)^2(2n-2)/\delta$.

If δ and d are fixed, then $k=O(1)$ and, by the prime number theorem for arithmetic progressions, $m=O(n)$. Thus F has $km/2=O(n)$ edges. To complete the proof of Theorem 2, we shall show that, if $D \subseteq E(F)$ with $|D| \geqq \delta km/2$, then the graph G with vertices $V(G)=V(F)$ and edges $E(G)=D$ contains an expanding graph.

Let $W \subseteq V(G)$ be a minimal non-empty set such that $|\Theta_G(W)| \geqq \delta k|W|/2$ (such a set exists, since $|\Theta_G(V(G))| \geqq \delta k|V(G)|/2$). Let H be the graph with vertices $V(H)=W$ and edges $E(H)=\Theta_G(W)$. We shall show that H is an expanding graph.

First, we claim that if $X \subseteq V(H)$, then $|\Phi_H(X)| \geqq \delta k|X|/2$. This is clear if $X=\emptyset$ or $X=W$. If $\emptyset \subset X \subset W$, then $W \setminus X$ is a proper non-empty subset of W, so by the minimality of W, $|\Theta_H(W \setminus X)| = |\Theta_G(W \setminus X)| < \delta k|W \setminus X|/2$. Thus $|\Phi_H(X)| = |\Theta_H(W)| - |\Theta_H(W \setminus X)| > \delta k|W|/2 - \delta k|W \setminus X|/2 = \delta k|X|/2$, and the claim is proved.

Now suppose that $X \subseteq V(H)$ with $|X| \leqq 2n-2$. We shall show that $|\Gamma_H(X)| \geqq (d+1)|X|$. Suppose, to obtain a contradiction, that $|\Gamma_H(X)| < (d+1)|X|$. Let $Y = X \cup \Gamma_H(X)$. Then $|Y| < (d+2)|X| \leqq (d+2)(2n-2)$. By Lemma 2.1 (with $l=$

$=2(k-1)^{1/2}\leqq 2k^{1/2})$ we have $|\Theta_H(Y)|\leqq|\Theta_F(Y)|\leqq k|Y|^2/2m+k^{1/2}|Y|$. By supposition we have $|\Theta_H(Y)|\leqq k(d+2)^2(2n-2)|X|/2m+k^{1/2})|Y|$, and by Condition 2, $|\Theta_H(Y)|\leqq \delta k|X|/4+k^{1/2}|Y|$. Since $\Phi_H(X)\subseteqq\Theta_H(Y)$, by the claim we have $|\Theta_H(Y)|\geqq \geqq\delta k|X|/2$. Thus $k^{1/2}|Y|\geqq\delta k|X|/4$, so $|Y|\geqq\delta k^{1/2}|X|/4$. By Condition 1, $k^{1/2}\geqq \geqq 4(d+2)/\delta$, so $|Y|\geqq(d+2)|X|$. This contradiction shows that H is an expanding graph and completes the proof of Theorem 2.

4. Proof of Theorem 3

We shall need the following lemma.

Lemma 3.1. *If a simple graph F has m vertices and is (k, l)-spectral, and if $X\subseteqq V(F)$, then*

$$|\Theta_F(X)| \geqq k|X|^2/2m-(l+2)|X|/2.$$

Proof. Since F is a simple graph, so is its complement $\bar{F}$, which has adjacency matrix $J-I-F$, where J is an $m\times m$ matrix with every entry equal to 1 and I is an $m\times m$ identity matrix. The graph $\bar{F}$ has eigenvalues $\lambda_0(\bar{F})=m-1-k$ and $\lambda_j(\bar{F})=-1-\lambda_j(F)$, $1\leqq j\leqq m-1$. Thus $\bar{F}$ is $(m-1-k, l+1)$-spectral. Applying Lemma 2.1 to $|\Theta_{\bar{F}}(X)|$ in the formula $|\Theta_F(X)|=|X|(|X|-1)/2-|\Theta_{\bar{F}}(X)|$ completes the proof. ∎

Let $\varepsilon>0$, d and n be given. Set $\delta=\varepsilon^2/2$. Assign p and q the smallest appropriate values for which Conditions 1 and 2 in the proof of Theorem 2 are satisfied and for which $m\geqq k^3$.

If ε and d are fixed, so is δ. Thus again $k=O(1)$ and $m=O(n)$, and the graph F has $km/2=O(n)$ edges. Furthermore, $m\geqq k^3$ ensures that the girth of F exceeds 2, so F is a simple graph. To complete the proof of Theorem 3, we shall show that, if $U\subseteqq V(F)$ with $|U|\geqq\varepsilon m$, then $|\Theta_F(U)|\geqq\delta km/2$, for then we may set $D= =\Theta_F(U)$ and apply Theorem 2.

Suppose, to obtain a contradiction, that $U\subseteqq V(F)$ with $|U|\geqq\varepsilon m$ but $|\Theta_F(U)|< <\delta km/2$. Since $\delta=\varepsilon^2/2$, $|\Theta_F(U)|<\varepsilon^2 km/4\leqq k|U|^2/4m$. By Lemma 3.1 (with $l+2= =2(k-1)^{1/2}+2\leqq 4k^{1/2}$) we have $|\Theta_F(U)|\geqq k|U|^2/2m-2k^{1/2}|U|$. Thus $2k^{1/2}|U|> >k|U|^2/4m$, so $|U|<8m/k^{1/2}$. By Condition 1, $k^{1/2}\geqq 4(d+2)/\delta\geqq 8/\varepsilon$, so $|U|<\varepsilon m$. This contradiction completes the proof of Theorem 3.

5. References

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